# Algebra III Final Exam

November 6 2015

This exam is of 50 marks. There are 8 questions with subquestions plus an extra credit question of $5$ marks. Please read all the questions carefully and do not cheat. Good luck! (50)

Unless otherwise stated $V$ will be a finite dimensional vector space over a field of characteristic $0$. $A_F(V) = \mathrm{End}_F(V) = \mathrm{Hom}_F(V, V)$.

1. Let $R$ be a ring. Let

$$Z(R) = \{x \in R | xy = yx \text{ for all } y \in R\}$$

Show

1. $Z(R)$ is a subring of $R$. (3)
2. If $R$ is a division ring, then $Z(R)$ is a field. (2)

2. If $\dim_F(V) > 1$, prove that $A_F(V)$ is not commutative. (5)

3 Let $V$ be a $2$-dimensional vector space with basis $\{e_1, e_2\}$. Let $T$ be defined by

$$T(e_1) = 7e_1 + 3e_2 \qquad T(e_2) = 3e_1 + 4e_2$$

Find:

1. The matrix representation of $T$ with respect to this basis. (2)
2. The eigenvalues of $T$. (3)
3. The eigenvectors of $T$. (3)
4. The characteristic polynomial of $T$ (3)
5. The minimal polynomial of $T$. (3)

4 Let $A, B$ be in $M_n(F)$, where $F$ is a field of characteristic $0$.

1. Show that $1 - (AB - BA)$ cannot be nilpotent. (3)

2. Is it necessarily still true if $\mathrm{char}(F) \neq 0$? (3)

5. Find the nilpotent canonical form of the matrix (5)

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

6. Find a matrix in $M_3(\mathbb{Q})$ satisfying the equation $X^3 + X + 1 = 0$. (5)
7. Find all possible Jordan forms of $3 \times 3$ matrices with minimal polynomial (5)

$$(X^3 - 6X^2 + 11X - 6)$$

8. Let $A \in M_n(F)$. Show, using the definition of the determinant, that $\det(A) = \det(A')$ where $A'$ is the transpose of $A$. (5)

9. **Extra Credit** Let $f$ be the polynomial (5)

$$f(X) = (X-1)(X-2)(X-3)(X-4)(X-5) + 1$$

Show $f$ is irreducible over $\mathbb{Q}$.